Solution exercise set 2

Problem 1

The Navier-Stokes equation for the incompressible flow of a Newtonian fluid reads

$$\rho\left(\frac{\partial \mathbf{v}}{\partial t}+(\mathbf{v}\cdot\nabla)\mathbf{v}\right)=-\nabla p+\mu\nabla^2\mathbf{v}+f \tag{1}$$

here f represents the gravity force per unit volume. For the stationary flow of the fluid in parallel layers (laminar flow), we have that $\mathbf{v}(x,y,z,t)=v(y)\mathbf{e}_x$ with $\mathbf{e}_x$ the unit vector along the flow direction. The differential equation for the velocity $v(y)$ of the two fluids reduces then to

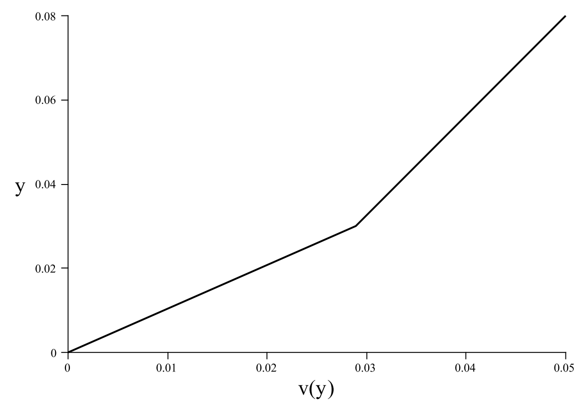
$$\frac{d^2v}{dy^2}=0. \tag{2}$$

To obtain (2) we used the fact that $\nabla^2\mathbf{v}$ is orthogonal to f and ∇p. Besides $(\mathbf{v}\cdot\nabla)\mathbf{v}=0$ (checking this is left as an exercise). Choosing the origin for the y axis at the lower plate, we get the boundary conditions supplementing equation (2)

$$\begin{aligned}
v(0) &= 0 && (3)\\
v(H_1+H_2) &= V_0 && (4)\\
\mu_1\frac{dv}{dy}|_{y=H_2+\epsilon} &= \mu_2\frac{dv}{dy}|_{y=H_2-\epsilon} \quad \text{for } \epsilon\to 0 && (5)\\
v(H_2+\epsilon) &= v(H_2-\epsilon) && (6)
\end{aligned}$$

the first two conditions are valid in the absence of a sliding boundary at the interface between the liquid and the plate and the last condition expresses the equality between the stresses at the boundary between the two fluids. The general solution to equation (2) can be written in the form

$$v(y)=\begin{cases} A_1y+B_1 & y>H_2\\ A_2y+B_2 & y<H_2\end{cases}$$



Note that we have $B_2 = 0$ to ensure that $v(0) = 0$. Using the boundary conditions we get algebraic equations for the other coefficients

$$\begin{cases} A_1(H_1 + H_2) + B_1 = V_0 \\ A_1 H_2 + B_1 - A_2 H_2 = 0 \\ \mu_1 A_1 - \mu_2 A_2 = 0. \end{cases}$$

Solving those equations gives us the final velocity profile $A_1 = 0.422$, $A_2 = 0.9628$ and $B_1 = 0.01624$.

Problem 2

This problem is solved in details in the textbook by Scott A. Socolofsky & Gerhard H. Jirka on page 11. The book can be found on Moodle.